

ON HECKE EIGENVALUES AT PIATETSKI-SHAPIRO PRIMES

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ABSTRACT. Let $\lambda(n)$ be the normalized n -th Fourier coefficient of a holomorphic cusp form for the full modular group. We show that for some constant $C > 0$ depending on the cusp form and every fixed c in the range $1 < c < 8/7$, the mean value of $\lambda(p)$ is $\ll \exp(-C\sqrt{\log N})$ as p runs over all (Piatetski-Shapiro) primes of the form $[n^c]$ with $n \in \mathbb{N}$ and $n \leq N$.

Mathematics Subject Classification (2000): 11F11, 11F30, 11F60, 11L03, 11L07, 11L20.

Keywords: Hecke eigenvalues, Piatetski-Shapiro primes

1. INTRODUCTION

Let f be a holomorphic cusp form of weight κ for the full modular group. By $\lambda_f(n)$ we denote the normalized n -th Fourier coefficient of f , *i.e.*

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(\kappa-1)/2} e(nz)$$

for $\Im z > 0$. The Ramanujan-Petersson conjecture, proved by P. Deligne [5, 6], states that $\lambda_f(n) \ll n^\varepsilon$ for any fixed $\varepsilon > 0$. More precisely, we have

$$(1.1) \quad \lambda_f(n) \ll d(n) \ll n^\varepsilon,$$

where $d(n)$ is the number of divisors of n . If we assume, in addition, that f is an eigenform of all the Hecke operators, then f can be normalized such that $\lambda_f(1) = 1$ and with this normalization the implied constant in the first “ \ll ” in (1.1) can be taken to be 1.

The distribution of Fourier coefficients of cusp forms has received a lot of attention. It is due to G. H. Hardy and S. Ramanujan that

$$\sum_{n \leq N} \lambda_f(n) e(\alpha n) \ll N^{1/2} \log(2N),$$

and it follows from a general formula of K. Chandrasekharan and N. Narasimhan [4] that

$$\sum_{n \leq N} \lambda_f(n) \ll N^{1/3+\varepsilon}.$$

It is worth noting that more recently N. J. E. Pitt [24] and V. Blomer [3] established, respectively, estimates for sums of the forms

$$\sum_{n \leq N} \lambda(n) e(\alpha n^2 + \beta n) \quad \text{and} \quad \sum_{n \leq N} \lambda(n^2 + sn + t),$$

with $\alpha, \beta \in \mathbb{R}$ and $s, t \in \mathbb{Z}$.

Especially interesting is the distribution of Fourier coefficients of cusp form at prime arguments. It is known that (see, for example, Section 5.6 of [13]) there exists a positive constant C , depending on the cusp form f , such that

$$(1.2) \quad \sum_{p \leq N} \lambda_f(p) \ll N \exp\left(-C\sqrt{\log N}\right),$$

where the implied \ll -constant depends on the cusp form f . Under the generalized Riemann hypothesis for modular L -functions, the right-hand side of (1.2) can be replaced by $N^{1/2+\varepsilon}$. M. R. Murty [22] conjectured an Ω -result of the form

$$\sum_{p \leq N} \lambda_f(p) = \Omega_{\pm} \left(\frac{\sqrt{N} \log \log \log N}{\log N} \right)$$

and succeeded in proving it provided some L -function has no real zero between $1/2$ and 1 . S. D. Adhikari [1] generalized this result to cusp forms for the group $\Gamma_0(N)$. The second-named author of the present paper investigated special exponential sums with Fourier coefficients of cusp forms over primes [29], motivated by some surprising heuristic due to H. Iwaniec, W. Luo and P. Sarnak [14] which gives that there should not be square-root cancellation in these exponential sums.

There is a more precise conjecture than (1.2) on the distribution of the $\lambda_f(p)$'s, known as the Sato-Tate conjecture. This conjecture states that if f is a primitive holomorphic cusp form of weight greater than 2 which is not of dihedral type, then the coefficients $\lambda_f(p)$ follow a certain distribution law. For the details, see [13], Chapter 21.

In the present paper, we investigate sums of Fourier coefficients of cusp forms over certain *sparse* sets of primes, namely Piatetski-Shapiro primes which we will discuss below. The motivation for our investigation is two-fold. First, the mean-values of arithmetic functions (in particular, of Fourier coefficients of cusp forms) over sparse sequences are often difficult to handle and thus of great interest. The work of V. Blomer [3] is in this direction. Second, it is a hard problem to detect primes in arithmetically interesting sets of natural numbers that are *sparse*. Recently, there has been much progress with regard to problems of this type. J. B. Friedlander and H. Iwaniec [9] established the celebrated result that there are infinitely many primes of the form $X^2 + Y^4$ with $X, Y \in \mathbb{N}$. D. R. Heath-Brown [12] proved the infinitude of the set of primes of the form $X^3 + 2Y^3$ with $X, Y \in \mathbb{N}$.

A classical result in the direction of finding primes in sparse sequences is the Piatetski-Shapiro prime number theorem which states that there exists $c > 1$ such that there are infinitely many primes of the form $[n^c]$ with $n \in \mathbb{N}$, where $[x]$ denotes the integral part of x . More precisely, I. I. Piatetski-Shapiro [23] proved that

$$(1.3) \quad |\{n \leq N : [n^c] \text{ is a prime number}\}| \sim \frac{N}{c \log N}, \text{ as } N \rightarrow \infty$$

if c is a fixed number lying in the range $1 < c < 12/11$. This c -range for which (1.3) holds has been widened by many authors (see [11, 18, 19, 21, 26]). The most recent record is due to J. Rivat and P. Sargos [25] who proved that (1.3) holds in the range $1 < c < 2817/2426$. Lower bounds of the correct order of magnitude for the quantity on the left-hand side of (1.3) were established by several authors (see [15, 16]) for wider c -ranges. It is conjectured that the asymptotic formula in (1.3) holds for all non-integers $c > 1$. (Note that for $0 < c \leq 1$, (1.3) follows easily from partial summation and the prime number theorem.) D. Leitman and D. Wolke [20] showed that (1.3) holds for almost all, with respect to Lebesgue measure, c with $1 < c < 2$. Moreover, it was due to J.-M. Deshouillers [7] that the left-hand side of (1.3) tends to infinity as n tends to infinity for almost all $c > 1$, with respect to Lebesgue measure.

The main result of this paper is the following.

Theorem 1. *Let $1 < c < 8/7$ and $\lambda_f(n)$ be the normalized n -th Fourier coefficient of a holomorphic cusp form f for the full modular group. Let \mathbb{P} denote the set of primes. Then there exists a constant $C > 0$ depending on f such that*

$$(1.4) \quad \sum_{\substack{n \leq N \\ [n^c] \in \mathbb{P}}} \lambda_f([n^c]) \ll N \exp \left(-C \sqrt{\log N} \right),$$

where the implied \ll -constant depends on c and the cusp form f .

We note that, by (1.3) and $8/7 = 1.142... < 1.161... = 2817/2426$ (recall that Sargos and Rivat established (1.3) for the range $1 < c < 2817/2426$), the right-hand side of (1.4) is small compared to the total number of Piatetski-Shapiro primes of the form $[n^c]$ with $n \leq N$ if c lies in the range given in Theorem 1.

An even harder problem is the question how the absolute values of the Fourier coefficients of cusp forms are distributed at Piatetski-Shapiro primes. For the full set of primes, one has the following results. If f is a normalized Hecke eigenform, then, similarly as the prime number theorem, it can be established by using the analytic properties of the Rankin-Selberg L -function $L(f \otimes \bar{f}, s)$ that

$$(1.5) \quad \sum_{p \leq N} |\lambda_f(p)|^2 \sim \frac{N}{\log N}.$$

For general cusp forms f , one has

$$(1.6) \quad \sum_{p \leq N} |\lambda_f(p)|^2 \sim c_f \frac{N}{\log N}$$

as $N \rightarrow \infty$, where c_f is some positive constant depending on f . To see this, write f as a linear combination of Hecke eigenforms and thus $\lambda_f(p)$ as a linear combination of the corresponding Fourier coefficients of these Hecke eigenforms, multiply out the modulus square, and use (1.5) together with the similarly established fact that

$$(1.7) \quad \sum_{p \leq N} \lambda_f(p) \overline{\lambda_g(p)} = o\left(\frac{N}{\log N}\right)$$

if f and g are linearly independent Hecke eigenforms. We conjecture that a result analogous to (1.6) holds for Piatetski-Shapiro primes.

Conjecture 1. *Under the assumptions of Theorem 1, there exists a constant $c_f > 0$ such that*

$$\sum_{\substack{n \leq N \\ [n^c] \in \mathbb{P}}} |\lambda_f([n^c])|^2 \sim c_f \frac{N}{c \log N}, \quad \text{as } N \rightarrow \infty.$$

If Conjecture 1 holds, then, using (1.1), we deduce that

$$\sum_{\substack{n \leq N \\ [n^c] \in \mathbb{P}}} |\lambda_f([n^c])| \gg \frac{N}{\log N}$$

which is large compared to the right-hand side of (1.4). This implies the following conditional result on the oscillations of Fourier coefficients of cusp forms at Piatetski-Shapiro primes.

Theorem 2. *Assume that the conditions of Theorem 1 are satisfied and Conjecture 1 holds. Then either $\Re \lambda_f(p)$ or $\Im \lambda_f(p)$ changes sign infinitely often at primes of the form $p = [n^c]$, $n \in \mathbb{N}$.*

In the following, we say some words about our method for the proof of Theorem 1. First, since every cusp form can be written as a linear combination of finitely many Hecke eigenforms, it will suffice to prove Theorem 1 for (normalized) Hecke eigenvalues. The advantages of working with Hecke eigenvalues are that they are multiplicative and real. Now for the proof of Theorem 1 with Hecke eigenvalues, we shall adapt parts of the method of [21] who established the validity of (1.3) for $1 < c < 15/13 = 1.153...$. Similarly as in [21] (c.f. also the paper [2] of the first-named author), we shall use estimates for certain trilinear exponential sums with monomials [8, 27]. However, the appearance of the Hecke eigenvalues shall require to introduce some new ingredients. In particular, we shall use a method of M. Jutila [17] to transform exponential sums of the form

$$\sum_{N_1 < n \leq N_2} \lambda_f(n) e(g(n))$$

into other exponential sums involving Hecke eigenvalues of different lengths. Jutila's method may be viewed as an analogue of the B-process in Weyl-van der Corput's method in the theory of exponential sums.

We note that the investigations in this paper lead to exponential sums that are closely related to those considered in [29]. However, the method used in [29] will not be appropriate for our purposes here, and, conversely, the method used in the present paper does not seem to lead to any improvement of the result in [29].

Notations. The following notations and conventions are used throughout the paper.

$$e(z) = \exp(2\pi iz) = e^{2\pi iz}.$$

η and ε are small positive real numbers, where ε may not be the same number in each occurrence.

$c > 1$ is a fixed number and we set $\gamma = 1/c$.

$\lambda(n)$ denotes the normalized n -th Fourier coefficients of a Hecke eigenform for the full modular group.

$\Lambda(n)$ is the van Mangoldt function.

$d(n)$ is the divisor function.

$k \sim K$ means $K_1 \leq k \leq K_2$ with $K/2 \leq K_1 \leq K_2 \leq 2K$.

$f = O(g)$ or $f \ll g$ means $|f| \leq cg$ for some unspecified positive constant c .

$f \asymp g$ means $f \ll g$ and $g \ll f$.

$[x]$ denotes the largest integer not exceeding x , and $\psi(x) = x - [x] - 1/2$ denotes the saw-tooth function.

2. PRELIMINARY LEMMAS

In this section, we quote the results needed later. To get started, we shall use the following approximation of the saw tooth function $\psi(x) = x - [x] - 1/2$ due to J. D. Vaaler [28].

Lemma 1 (Vaaler). *For $0 < |t| < 1$, let*

$$W(t) = \pi t(1 - |t|) \cot \pi t + |t|.$$

Fix a positive integer J . For $x \in \mathbb{R}$ define

$$\psi^*(x) := - \sum_{1 \leq |j| \leq J} (2\pi i j)^{-1} W\left(\frac{j}{J+1}\right) e(jx)$$

and

$$\delta(x) := \frac{1}{2J+2} \sum_{|j| \leq J} \left(1 - \frac{|j|}{J+1}\right) e(jx).$$

Then δ is non-negative, and we have

$$|\psi^*(x) - \psi(x)| \leq \delta(x)$$

for all real numbers x .

Proof. This is Theorem A6 in [10] and has its origin in [28]. □

We shall also use the following estimate for a sum involving the function δ .

Lemma 2. *Fix $0 < \gamma < 1$. Assume that $1 \leq N < N_1 \leq 2N$. Define the function δ as in Lemma 1. Then*

$$\sum_{N < n \leq N_1} \delta(-n^\gamma) \ll J^{-1}N + J^{1/2}N^{\gamma/2}.$$

Proof. This was proved on page 48 in [10]. □

We shall also need the following variant of the prime number theorem for Hecke eigenvalues which is equivalent to (1.2).

Lemma 3. *There exists a positive constant C , such that*

$$\sum_{n \leq N} \Lambda(n) \lambda(n) \ll N \exp\left(-C\sqrt{\log N}\right),$$

where the implied \ll -constant and the constant C depend on the cusp form.

Proof. This is a special case of the more general Theorem 5.12 in [13]. □

We shall then see that it suffices to prove that

$$(2.1) \quad \sum_{n \sim N} \Lambda(n) f(n) = O(N^{1-\eta})$$

for a some fixed $\eta > 0$, where f is a certain function involving the Hecke eigenvalue $\lambda(n)$ and a trigonometric polynomial. The following lemma reduces the above sum containing the von Mangoldt function to so-called type I and type II sums.

Lemma 4 (Heath-Brown). *Let f be a complex-valued function defined on the natural numbers. Suppose that u, v and z are real parameters satisfying the conditions*

$$3 \leq u < v < z < 2N, \quad z - 1/2 \in \mathbb{N}, \quad z \geq 4u^2, \quad N \geq 32z^2u, \quad v^3 \geq 64N.$$

Suppose further that $1 \leq Y \leq N$ and $XY = N$. Assume that a_m and b_n are complex numbers. We write

$$(2.2) \quad K := \sum_{\substack{m \sim X \\ mn \sim N}} \sum_{\substack{n \sim Y}} a_m f(mn)$$

and

$$(2.3) \quad L := \sum_{\substack{m \sim X \\ mn \sim N}} \sum_{\substack{n \sim Y}} a_m b_n f(mn).$$

Then the estimate (2.1) holds if we uniformly have

$$K \ll N^{1-2\eta} \quad \text{for } Y \geq z \text{ and any complex } a_m \ll 1$$

and

$$L \ll N^{1-2\eta} \quad \text{for } u \leq Y \leq v \text{ and any complex } a_m, b_n \ll 1.$$

Proof. This is a consequence of Lemma 3 in [11]. □

To separate the variables m and n appearing in the previous Lemma 4, we shall use the following lemmas. The first of them is the multiplicative property of Hecke eigenvalues, and the second of them is a variant of Perron's formula.

Lemma 5. *Hecke eigenvalues are multiplicative and they satisfy the following relation.*

$$\lambda(mn) = \sum_{d \mid \gcd(m, n)} \mu(d) \lambda\left(\frac{m}{d}\right) \lambda\left(\frac{n}{d}\right).$$

Proof. This Lemma follows by applying the Möbius inversion formula to the product formula for the Hecke eigenvalues. See, for example, Proposition 14.9 of [13]. □

Lemma 6. *Let $0 < M \leq N < \nu N < \kappa M$ and let a_m be complex numbers with $|a_m| \leq 1$. We then have*

$$(2.4) \quad \sum_{N < n < \nu N} a_n = \frac{1}{2\pi} \int_{-M}^M \left(\sum_{M < m < \kappa M} a_m m^{-it} \right) N^{it} (\nu^{it} - 1) t^{-1} dt + O(\log(2 + M)),$$

where the implied O -constant depends only on κ .

Proof. This is Lemma 6 in [8]. □

We shall be led to certain trilinear exponential sums with monomials. A part of them shall be estimated by using the following bound due to O. Robert and P. Sargos [27] which is a sharpening of an earlier estimate of E. Fouvry and H. Iwaniec [8].

Lemma 7 (Robert and Sargos). *Let $\alpha, \alpha_1, \alpha_2$ be real constants such that $\alpha \neq 1, \alpha\alpha_1\alpha_2 \neq 0$. Let $M, M_1, M_2, x \geq 1$ and $|\phi_m| \leq 1, |\psi_{m_1, m_2}| \leq 1$. Then we have*

$$\begin{aligned} & \sum_{m \sim M} \sum_{m_1 \sim M_1} \sum_{m_2 \sim M_2} \phi_m \psi_{m_1, m_2} e \left(x \frac{m^\alpha m_1^{\alpha_1} m_2^{\alpha_2}}{M^\alpha M_1^{\alpha_1} M_2^{\alpha_2}} \right) \\ & \ll \left(x^{1/4} M^{1/2} (M_1 M_2)^{3/4} + M^{1/2} M_1 M_2 + M (M_1 M_2)^{3/4} + x^{-1/2} M M_1 M_2 \right) (M M_1 M_2)^\varepsilon. \end{aligned}$$

Proof. This follows from Theorem 1 in [27]. \square

To transform exponential sums of the form $\sum_n \lambda(n) e(g(n))$ into other exponential sums involving Hecke eigenvalues, we shall utilize Jutila's result [17] quoted below.

Lemma 8 (Jutila). *Let $\delta_1, \delta_2, \dots$ denote positive constants which may be supposed to be arbitrarily small. Further, let $M_1 \geq 2$ and put $L = \log M_1$. Let $a(n)$ be the n -th Fourier coefficient of a holomorphic cusp form f for the full modular group, i.e.*

$$f(z) = \sum_{n=1}^{\infty} a(n) e(nz)$$

for $\Im z > 0$. Let κ be the weight of the cusp form f . Let $M_1 < M_2 \leq 2M_1$, and let g and w be holomorphic functions in the domain

$$D = \{z : |z - x| < cM_1 \text{ for some } x \in [M_1, M_2]\},$$

where c is a positive constant. Suppose that $g(x)$ is real for $M_1 \leq x \leq M_2$. Suppose also that, for some positive numbers G and W ,

$$(2.5) \quad |w(z)| \ll W \quad \text{and} \quad |g'(z)| \ll GM_1^{-1} \quad \text{for } z \in D,$$

and that

$$(2.6) \quad (0 <) g''(x) \gg GM_1^{-2} \quad \text{for } M_1 \leq x \leq M_2.$$

Let $r = l/k$ with $(l, k) = 1$ be a rational number such that

$$(2.7) \quad 1 \leq k \ll M_1^{1/2-\delta_1}, \quad |r| \asymp GM_1^{-1} \quad \text{and} \quad g'(M_0) = r$$

for a certain number $M_0 \in (M_1, M_2)$. Write

$$(2.8) \quad M_j = M_0 + (-1)^j m_j, \quad j = 1, 2.$$

Suppose that $m_1 \asymp m_2$, and that

$$(2.9) \quad M_1^{\delta_2} \max \left\{ M_1 G^{-1/2}, |lk| \right\} \ll m_1 \ll M_1^{1-\delta_3}.$$

Define for $j = 1, 2$

$$p_{j,n} = g(x) - rx + (-1)^{j-1} \left(\frac{2\sqrt{nx}}{k} - \frac{1}{8} \right) \quad \text{and} \quad n_j = (r - g'(M_j))^2 k^2 M_j,$$

and for $n < n_j$ let $x_{j,n}$ be the (unique) zero of $p'_{j,n}(x)$ in the interval (M_1, M_2) . Then

$$\begin{aligned} & \sum_{M_1 \leq m \leq M_2} a(m) w(m) e(g(m)) \\ (2.10) \quad & = \frac{i}{\sqrt{2k}} \sum_{j=1}^2 (-1)^{j-1} \sum_{n < n_j} a(n) e \left(-\frac{n\bar{l}}{k} \right) n^{-\kappa/2+1/4} x_{j,n}^{\kappa/2-3/4} \frac{w(x_{j,n})}{\sqrt{p''_{j,n}(x_{j,n})}} e \left(p_{j,n}(x_{j,n}) + \frac{1}{8} \right) \\ & \quad + O \left(W(|l|k)^{1/2} M_1^{(\kappa-1)/2} m_1^{1/2} L^2 + G^{1/2} W |l|^{-3/4} k^{5/4} M_1^{(\kappa-1)/2} m_1^{-1/4} L \right), \end{aligned}$$

where $\bar{l} \equiv 1 \pmod{k}$.

Proof. This is Theorem 3.2 in [17] with different notations. \square

Lemma 8 lies at the heart of our method. Using this result, Jutila [17] proved the following estimate for “long” exponential sums with Hecke eigenvalues.

Lemma 9 (Jutila). *Let $t \geq 1$ and $M \geq 2$. Assume that*

$$(2.11) \quad M^{3/4-\gamma} \ll t \ll M^{3/2-\gamma}.$$

Then

$$(2.12) \quad \sum_{n \sim M} \lambda(n) e(tn^\gamma) \ll t^{1/3} M^{1/2+\gamma/3} (tM)^\varepsilon.$$

Proof. This follows from Theorem 4.6 in [17]. \square

The above Lemma 9 is not needed in our method. We shall rather use Lemma 8 on short exponential sums with Hecke eigenvalues. However, in section 6 we will see that a direct application of Lemma 9 also leads to a non-trivial result. Yet, this result is weaker than our main result, Theorem 1.

We also use the following lemma in the investigation of the spacing of certain monomial points.

Lemma 10. *Let $\alpha\beta \neq 0$, $\Delta > 0$, $M \geq 1$ and $N \geq 1$. Let $\mathcal{A}(M, N; \Delta)$ be the number of quadruples $(m, \tilde{m}, n, \tilde{n})$ such that*

$$\left| \left(\frac{\tilde{m}}{m} \right)^\alpha - \left(\frac{\tilde{n}}{n} \right)^\beta \right| < \Delta,$$

with $M \leq m, \tilde{m} < 2M$ and $N \leq n, \tilde{n} < 2N$. We then have

$$\mathcal{A}(M, N; \Delta) \ll MN \log 2MN + \Delta M^2 N^2.$$

Proof. This is Lemma 1 in [8]. \square

Finally, we need the following classical exponential sum estimate.

Lemma 11 (Van der Corput). *Let $b - a \geq 1$ and let $f(x)$ be a twice differentiable function on (a, b) such that $\Lambda \leq |f''(x)| \leq \nu\Lambda$, where $\Lambda > 0$ and $\nu \geq 1$. Then*

$$\sum_{a < n \leq b} e(f(n)) \ll \nu\Lambda^{1/2}(b-a) + \Lambda^{-1/2}.$$

Proof. This is Lemma 4.1 in [17]. \square

3. REDUCTION TO EXPONENTIAL SUMS

Using (1.1), partial summation, and the fact that every cusp form can be written as a linear combination of finitely many Hecke eigenforms, Theorem 1, our main result, can be easily deduced from the following result whose proof will be the object of the remainder of this paper.

Theorem 3. *Let $1 < c < 8/7$ and $\lambda(n)$ be the normalized n -th Fourier coefficient of a Hecke eigenform for the full modular group. By $\Lambda(n)$ denote the von Mangoldt function. Then there exists a positive constant C depending on the cusp form such that*

$$(3.1) \quad \sum_{n \leq N} \Lambda([n^c]) \lambda([n^c]) \ll N \exp(-C\sqrt{\log N}),$$

where the implied \ll -constant depends only on c , C and the cusp form.

In this section, we reduce the left-hand side of (3.1) to exponential sums. Throughout the sequel, let $\gamma = 1/c$. Then $[n^c] = m$ is equivalent to

$$-(m+1)^\gamma < -n \leq -m^\gamma.$$

Therefore, we have

$$(3.2) \quad \sum_{n \leq N} \Lambda([n^c]) \lambda([n^c]) = \sum_{m \leq N^c} ([-m^\gamma] - [-(m+1)^\gamma]) \Lambda(m) \lambda(m) + O(\log N).$$

Breaking into dyadic intervals, it hence suffices to prove that

$$(3.3) \quad S := \sum_{n \sim N^c} ([-n^\gamma] - [-(n+1)^\gamma]) \Lambda(n) \lambda(n) \ll N \exp(-C\sqrt{\log N})$$

for any $N > 1$. We write the above sum S in the form

$$(3.4) \quad S = S_1 + S_2,$$

where

$$S_1 = \sum_{n \sim N^c} ((n+1)^\gamma - n^\gamma) \Lambda(n) \lambda(n)$$

and

$$S_2 = \sum_{n \sim N^c} (\psi(-(n+1)^\gamma) - \psi(-n^\gamma)) \Lambda(n) \lambda(n),$$

with $\psi(n)$ being the saw-tooth function in Lemma 1. Using partial summation and the bounds

$$(x+1)^\gamma - x^\gamma \ll x^{\gamma-1} \quad \text{and} \quad \frac{d}{dx}((x+1)^\gamma - x^\gamma) \ll x^{\gamma-2}$$

for $x \geq 1$, we deduce from Lemma 3 that

$$S_1 \ll N \exp(-C\sqrt{\log N}),$$

where the implied \ll -constant depends only on γ , C and the cusp form. Our treatment of the sum S_2 begins like in [10]. By Lemma 1, we have the following. For any $J > 0$ there exist functions ψ^* and δ , with δ non-negative, such that

$$\psi(x) = \psi^*(x) + O(\delta(x)),$$

where

$$\psi^*(x) = \sum_{1 \leq |j| \leq J} a(j) e(jx), \quad \delta(x) = \sum_{|j| \leq J} b(j) e(jx)$$

with

$$a(j) \ll j^{-1}, \quad b(j) \ll J^{-1}.$$

Consequently,

$$\begin{aligned} S_2 &= \sum_{n \sim N^c} (\psi^*(-(n+1)^\gamma) - \psi^*(-n^\gamma)) \Lambda(n) \lambda(n) + O\left((\log N) \sum_{n \sim N^c} (\delta(-(n+1)^\gamma) + \delta(-n^\gamma))\right) \\ &= S_3 + O(S_4), \end{aligned}$$

say. We fix a small $\eta > 0$ and set

$$(3.5) \quad J := N^{c-1+\eta}.$$

Then, using Lemma 2, we obtain

$$S_4 \ll N^{1-\eta/2}$$

if $1 < c < 2$.

The remaining task is to prove that

$$S_3 \ll N^{1-\eta/2},$$

provided that η is sufficiently small. We write

$$S_3 = \sum_{1 \leq |j| \leq J} \sum_{n \sim N^c} \Lambda(n) \lambda(n) a(j) \phi_j(n) e(-jn^\gamma),$$

where $\phi_j(x) = 1 - e(j(x^\gamma - (x+1)^\gamma))$. Using partial summation and the bounds

$$\phi_j(x) \ll jx^{\gamma-1} \quad \text{and} \quad \frac{d}{dx} \phi_j(x) \ll jx^{\gamma-2},$$

we deduce that it suffices to prove that

$$\sum_{1 \leq |j| \leq J} \left| \sum_{n \sim N^c} \Lambda(n) \lambda(n) e(-jn^\gamma) \right| \ll N^{c-\eta/2}.$$

Replacing N^c by N , taking the definition of J in (3.5) into account, dividing the summation interval $1 \leq |j| \leq J$ into $O(\log 2J)$ dyadic intervals, and using the facts that $e(-x) = \overline{e(x)}$ and the Hecke eigenvalues are real, we see that the above bound holds if

$$(3.6) \quad \sum_{h \sim H} \left| \sum_{n \sim N} \Lambda(n) \lambda(n) e(hn^\gamma) \right| \ll N^{1-\eta}$$

for any $N \geq 1$ and $1 \leq H \leq N^{1-\gamma+\eta}$. The following lemma reduces the term on the left-hand side of (3.6) to trilinear exponential sums.

Lemma 12. *Suppose that u, v and z are real parameters satisfying the conditions*

$$(3.7) \quad 3 \leq u < v < z < 2N, \quad z - 1/2 \in \mathbb{N}, \quad z \geq 4u^2, \quad N \geq 32z^2u, \quad v^3 \geq 64N.$$

Suppose further that $1 \leq Y \leq N$, $XY = N$ and $H \geq 1$. Assume that A_m, B_n and C_h are complex numbers. For $d \in \mathbb{N}$ set

$$(3.8) \quad K_d := \sum_{\substack{m \sim X/d \\ mn \sim N/d^2}} \sum_{n \sim Y/d} \sum_{h \sim H} A_m C_h \lambda(n) e(hd^{2\gamma} m^\gamma n^\gamma)$$

and

$$(3.9) \quad L_d := \sum_{\substack{m \sim X/d \\ mn \sim N/d^2}} \sum_{n \sim Y/d} \sum_{h \sim H} A_m B_n C_h e(hd^{2\gamma} m^\gamma n^\gamma).$$

Then the estimate (3.6) holds if we uniformly have

$$(3.10) \quad K_d \ll N^{1-3\eta} d^{-1} \quad \text{for } Y \geq z, \quad d \leq 2Y \text{ and any complex } A_m, C_h \ll 1$$

and

$$(3.11) \quad L_d \ll N^{1-3\eta} d^{-1} \quad \text{for } u \leq Y \leq v, \quad d \leq 2Y \text{ and any complex } A_m, B_n, C_h \ll 1.$$

Proof. We first write

$$\sum_{h \sim H} \left| \sum_{n \sim N} \Lambda(n) \lambda(n) e(hn^\gamma) \right| = \sum_{h \sim H} c_h \sum_{n \sim N} \Lambda(n) \lambda(n) e(hn^\gamma),$$

where c_h are suitable complex numbers with $|c_h| = 1$. We further set

$$f(n) = \lambda(n) \sum_{h \sim H} c_h e(hn^\gamma)$$

so that

$$\sum_{h \sim H} \left| \sum_{n \sim N} \Lambda(n) \lambda(n) e(hn^\gamma) \right| = \sum_{n \sim N} \Lambda(n) f(n).$$

Now, by Lemma 4, the bound (3.6) holds if

$$(3.12) \quad K \ll N^{1-2\eta} \quad \text{and} \quad L \ll N^{1-2\eta}$$

under the conditions of the same lemma. Here K and L are defined as in (2.2) and (2.3). We may rewrite these terms in the form

$$K = \sum_{\substack{m \sim X \\ mn \sim N}} \sum_{n \sim Y} \sum_{h \sim H} a_m c_h \lambda(mn) e(h(mn)^\gamma)$$

and

$$L = \sum_{\substack{m \sim X \\ mn \sim N}} \sum_{n \sim Y} \sum_{h \sim H} a_m b_n c_h \lambda(mn) e(h(mn)^\gamma).$$

Using the multiplicative property of Hecke eigenvalues, Lemma 5, we have

$$(3.13) \quad K = \sum_{d \leq 2Y} \mu(d) \sum_{\substack{m \sim X/d \\ mn \sim N/d^2}} \sum_{n \sim Y/d} \sum_{h \sim H} a_{dm} \lambda(m) c_h \lambda(n) e(hd^{2\gamma} m^\gamma n^\gamma)$$

and

$$(3.14) \quad L = \sum_{d \leq 2Y} \mu(d) \sum_{\substack{m \sim X/d \\ mn \sim N/d^2}} \sum_{n \sim Y/d} \sum_{h \sim H} a_{dm} \lambda(m) b_{dn} \lambda(n) c_h e(hd^{2\gamma} m^\gamma n^\gamma).$$

Now, (3.12) follows from (3.10), (3.11), (3.13), (3.14) and the bound $\lambda(n) \ll n^\varepsilon$, the Ramanujan-Petersson conjecture, proved by Deligne [5]. \square

In the following sections, we shall estimate the terms K_d and L_d .

4. ESTIMATION OF L_d

In this section, we estimate L_d defined in (3.9).

Lemma 13. *Let Q be any positive integer and ε be any positive real number. Then*

$$(4.1) \quad |L_d|^2 \ll \left(QX (HX^\gamma Y^\gamma Q^{-1})^{1/2} (H^2 Q^{-1} Y^2 + HY) + QX^{2-\gamma} (HY^{2-\gamma} + HYX^\gamma) \right) d^{-2} N^\varepsilon,$$

where the implied \ll -constant depends only on ε .

Proof. This lemma follows after a slight modification of the estimations in section 4 of [11]. \square

From Lemma 13, we deduce the following result.

Lemma 14. *If $H, N \geq 1$, $1 \leq Y \leq 2N$ and $1 \leq d \leq 2Y$, then*

$$(4.2) \quad |L_d|^2 \ll (N^{1+\gamma/2} H^{3/2} + N^{2-\gamma} H + N^2 H Y^{-1} + N^{4/3+\gamma/3} H^2 Y^{1/3} + N^{2/3+2\gamma/3} H^2 Y^{2/3} + N^{4/3-2\gamma/3} H^2 Y^{4/3}) d^{-2} N^\varepsilon,$$

where the implied \ll -constant depends only on ε .

Proof. To optimize the estimate (4.1) in Lemma 13, we choose

$$Q := 1 + \left[H X^{(\gamma-2)/3} Y^{(\gamma+2)/3} \right].$$

Then (4.2) follows from (4.1) and $XY = N$ by a short calculation. \square

Lemma 14 brings us into a position to formulate a condition under which the desired estimate $L_d \ll N^{1-3\eta} d^{-1}$ holds.

Lemma 15. *For every sufficiently small fixed $\eta > 0$, we have*

$$L_d \ll N^{1-3\eta} d^{-1},$$

provided that $\gamma > 5/6$, $1 \leq H \leq N^{1-\gamma+\eta}$, $1 \leq d \leq 2Y$ and

$$(4.3) \quad N^{1-\gamma+100\eta} \leq Y \leq N^{5\gamma-4-100\eta}.$$

Proof. This Lemma follows from Lemma 14 by a short calculation and is analogous to Lemma 4 in [11]. \square

5. ESTIMATION OF K_d , FIRST METHOD

We now establish some estimates for K_d , defined in (3.8), which are favorable if Y is not too large. In this case, we ignore the special nature of the Hecke eigenvalues $\lambda(n)$ appearing in the sum K_d and treat K_d as a trilinear sum with arbitrary coefficients, like L_d in the previous section.

For Y of medium size, we use the following result.

Lemma 16. *For every sufficiently small fixed $\eta > 0$, we have*

$$K_d \ll N^{1-3\eta} d^{-1},$$

provided that $\gamma > 5/6$, $1 \leq H \leq N^{1-\gamma+\eta}$, $1 \leq d \leq 2Y$ and

$$(5.1) \quad N^{5-5\gamma+100\eta} \leq Y \leq N^{\gamma-100\eta}.$$

Proof. This can be proved in essentially the same way as Lemma 15, where the roles of X and Y are reversed. Similarly as in Lemma 15, we get that $K_d \ll N^{1-3\eta}d^{-1}$, provided that

$$N^{1-\gamma+100\eta} \leq X \leq N^{5\gamma-4-100\eta}.$$

This is equivalent to (5.1) since $XY = N$. \square

If Y is small, then, similarly as in [21], we can directly apply Lemma 7 to estimate the term K_d defined in (3.8). This gives the following result.

Lemma 17. *If $H, N \geq 1$, $N^\eta H \leq Y \leq 2N$ and $1 \leq d \leq 2Y$, then*

$$(5.2) \quad K_d \ll d^{-1} \left(N^{3/4+\gamma/4} H Y^{-1/4} + N H Y^{-1/2} + N^{3/4} H^{3/4} Y^{1/4} + N^{1-\gamma/2} H^{1/2} \right) (NH)^\varepsilon.$$

Proof. First, we remove the summation condition $mn \sim N/d^2$ on the right-hand side of (3.8) by using Lemma 6 and thus make the summation ranges of m and n independent. After applying the bound $\lambda(n) \ll n^\varepsilon$ (Ramanujan-Petersson conjecture), the first term on the right-hand side of (2.4) leads to expressions of the form

$$N^\varepsilon \sum_{m \sim X/d} \sum_{n \sim Y/d} \sum_{h \sim H} \phi_m \psi_n \epsilon_h e(hd^{2\gamma} m^\gamma n^\gamma)$$

with $|\phi_m|, |\psi_n|, |\epsilon_h| \leq 1$. We then estimate these trilinear sums by applying Lemma 7 with the following choice of parameters:

$$x := HN^\gamma, \quad M := \frac{Y}{d}, \quad M_1 := \frac{X}{d}, \quad M_2 := H, \quad \alpha := \gamma, \quad \alpha_1 := \gamma, \quad \alpha_2 := 1.$$

Additionally taking $XY = N$ into account, we arrive at the estimate (5.2) upon noting that the contribution of the O -term on the right-hand side of (2.4) is $\ll N^{1+\varepsilon} H Y^{-1} d^{-1}$ and thus negligible by the condition $Y \geq N^\eta H$ in the lemma. \square

Lemma 17 enables us to formulate another condition under which the desired estimate $K_d \ll N^{1-3\eta}d^{-1}$ holds.

Lemma 18. *For every sufficiently small fixed $\eta > 0$, we have*

$$K_d \ll N^{1-3\eta}d^{-1},$$

provided that $\gamma > 5/6$, $1 \leq H \leq N^{1-\gamma+\eta}$, $1 \leq d \leq 2Y$ and

$$N^{3-3\gamma+100\eta} \leq Y \leq N^{3\gamma-2-100\eta}.$$

Proof. This follows from Lemma 17 by a short calculation. \square

We note that Lemma 18 could also be established by using the original bound of E. Fouvry and H. Iwaniec for trilinear exponential sums with monomials, Theorem 3 in [8], but with the more restrictive condition $\gamma > 17/20$ in place of $\gamma > 5/6$. Moreover, we would need to apply Theorem 3 in [8] twice with different choices of M , M_1 and M_2 which would complicate the computations.

6. SPLITTING OF THE SUM K_d

We now turn to the case when Y is large in which the special nature of the Hecke eigenvalues $\lambda(n)$ will become important.

A natural idea would be to apply Jutila's estimate for "long" exponential sums with Hecke eigenvalues in Lemma 9 directly to the sum over n in K_d and then to sum up over h and m trivially. But it turns out that this leads to the condition $Y \gg N^{8/3-2\gamma+100\eta}$ which is not sufficient to establish the c -range $1 < c < 8/7$ in Theorem 1. In order to obtain the desired estimate $K_d \ll N^{1-3\eta}d^{-1}$ for all relevant Y 's, we would need that $\gamma > 8/9$ which means that we would only get the c -range $1 < c < 9/8 = 1.125$ in Theorem 1. This is due to the fact that $N^{8/3-2\gamma+100\eta}$ is larger than the term $N^{\gamma-100\eta}$ in (5.1) whenever $\gamma \leq 8/9$.

To obtain the desired estimate for K_d in an as large as possible Y -range, we proceed as follows. First, following Jutila [17], we split the sum involving Hecke eigenvalues over n into shorter sums which we then

transform into new exponential sums with Hecke eigenvalues by applying Lemma 8 due to Jutila. Collecting all terms, we arrive at multi-linear exponential sums. To estimate them, we refine Jutila's treatment of long exponential sums with Hecke eigenvalues in [17]. Here we take advantage of the additional summations over h and m . This will lead to a spacing problem with certain points depending on h , m and further integers. We shall show that these points are essentially distributed as expected.

We first make some observations on Farey sequences. Let $K \geq 1$. By $\mathcal{F}(K)$, we denote the extended sequence of Farey fractions of level K consisting of all fractions of the form l/k , $1 \leq k \leq K$, $\gcd(l, k) = 1$. For two consecutive Farey fractions l/k and l'/k' in the sequence $\mathcal{F}(K)$, define the mediant, ρ , to be

$$(6.1) \quad \rho\left(\frac{l}{k}, \frac{l'}{k'}\right) = \frac{l+l'}{k+k'}.$$

Furthermore, if $l''/k'' < l/k < l'/k'$ are three consecutive Farey fractions, then we define the Farey interval (depending on K) around l/k by

$$(6.2) \quad I\left(\frac{l}{k}\right) = \left(\rho\left(\frac{l''}{k''}, \frac{l}{k}\right), \rho\left(\frac{l}{k}, \frac{l'}{k'}\right)\right].$$

We note that the set of the real numbers is the disjoint union of all these Farey intervals. Under the above notations, we further define $A_j(l/k)$ for $j = 1, 2$ by

$$\left(\frac{l}{k} - \frac{A_1(l/k)}{kK}, \frac{l}{k} + \frac{A_2(l/k)}{kK}\right] = I\left(\frac{l}{k}\right).$$

We note that (see, for example, (4.2.11) in [17])

$$(6.3) \quad A_j\left(\frac{l}{k}\right) \asymp 1.$$

For the proof of the desired bound $K_d \ll N^{1-3\eta}/d$, it suffices, by partial summation, the realness of $\lambda(n)$ and the fact $e(-x) = \overline{e(x)}$, to prove that

$$(6.4) \quad \tilde{K}_d \ll \frac{N^{1-3\eta}}{d} \left(\frac{Y}{d}\right)^{\frac{\kappa-1}{2}},$$

where

$$(6.5) \quad \tilde{K}_d := \sum_{h \sim H} \sum_{m \sim X/d} \sum_{\substack{Y_1/d < n \leq Y_2/d \\ N_1/(d^2 m) < n \leq N_2/(d^2 m)}} A_m C_h a(n) e(-hd^{2\gamma} m^\gamma n^\gamma).$$

Here

$$a(n) = \lambda(n) n^{\frac{\kappa-1}{2}}$$

is the un-normalized Fourier coefficient of the cusp form and

$$Y_j \asymp Y \text{ and } N_j \asymp N, \text{ for } j = 1, 2.$$

We prefer to have a negative sign in the e -term on the right-hand side of (6.5) for technical reasons.

We describe here briefly what we will do in the remainder of the section. We shall split the sum \tilde{K}_d into a sum of short exponential sums to which a Jutila-type transformation lemma (Lemma 19) can be applied. To this end, we cut the summation over n into small pieces so that the value of the derivative of the amplitude function in (6.5) on each of the small pieces is close to a fraction l/k whose denominator is not too large. In this treatment, we may incur an error which comes from the possible imperfect fit of the “end-intervals” in the splitting. This error will be estimated.

For $d \in \mathbb{N}$, $h \sim H$, $m \sim X/d$, $l < 0$ and $1 \leq k \leq K$, we define

$$(6.6) \quad g_{d,h,m}(x) = -hd^{2\gamma} m^\gamma x^\gamma$$

and $M_j(d, h, m; l/k)$ for $j = 0, 1, 2$ by

$$(6.7) \quad g' \left(M_0 \left(d, h, m; \frac{l}{k} \right) \right) = \frac{l}{k}, \text{ and } g' \left(M_j \left(d, h, m; \frac{l}{k} \right) \right) = \frac{l}{k} + (-1)^j \frac{A_j(l/k)}{kK}.$$

We further set

$$\mathcal{J} \left(d, h, m; \frac{l}{k} \right) = \left(M_1 \left(d, h, m; \frac{l}{k} \right), M_2 \left(d, h, m; \frac{l}{k} \right) \right]$$

and

$$\mathcal{I}(d, m) = \left(\frac{Y_1}{d}, \frac{Y_2}{d} \right] \cap \left(\frac{N_1}{d^2 m}, \frac{N_2}{d^2 m} \right].$$

As mentioned above, we shall approximate \tilde{K}_d by

$$(6.8) \quad K_d^* = \sum_{h \sim H} \sum_{m \sim X/d} C_h A_m \sum_{1 \leq k \leq K} \sum_{\substack{l < 0, \gcd(l, k) = 1 \\ M_0(d, h, m; l/k) \in \mathcal{I}(d, m)}} \sum_{n \in \mathcal{J}(d, h, m; l/k)} a(n) e(-hd^2 \gamma m^\gamma n^\gamma).$$

We now estimate the error of this approximation. For every h and m , there are at most two fractions of the form l/k with $1 \leq k \leq K$ and $\gcd(l, k) = 1$ such that the interval $\mathcal{J}(d, h, m; l/k)$ is not contained in the interval $\mathcal{I}(d, m)$ but over-laps with $\mathcal{I}(d, m)$. The contribution arising from an interval $\mathcal{J}(d, h, m; l/k)$ of this kind to the inner-triple sum on the right-hand side of (6.8) is

$$\ll \left(\frac{Y}{d} \right)^{\frac{\kappa-1}{2} + \varepsilon} \left| \mathcal{J} \left(d, h, m; \frac{l}{k} \right) \right|.$$

Using (6.3), it is easy to compute, with h, m, k and l subject to the same conditions as those in the summations in (6.8), that the length of $\mathcal{J}(d, h, m; l/k)$ is

$$(6.9) \quad \left| \mathcal{J} \left(d, h, m; \frac{l}{k} \right) \right| \ll \frac{Y^{2-\gamma}}{H d^2 X^\gamma k K}$$

(compare with (7.14)). Thus, the error in approximating \tilde{K}_d by K_d^* is

$$(6.10) \quad \tilde{K}_d - K_d^* \ll H \frac{X}{d} \left(\frac{Y}{d} \right)^{\frac{\kappa-1}{2} + \varepsilon} \frac{Y^{2-\gamma}}{H d^2 X^\gamma K} \ll N^{1-\gamma+\varepsilon} \frac{Y}{K d^3} \left(\frac{Y}{d} \right)^{\frac{\kappa-1}{2}},$$

where we use $XY = N$. This above error is negligible, i.e.

$$(6.11) \quad \tilde{K}_d - K_d^* \ll \frac{N^{1-3\eta}}{d} \left(\frac{Y}{d} \right)^{\frac{\kappa-1}{2}},$$

if

$$(6.12) \quad K \gg N^{4\eta} \frac{Y}{N^\gamma d^2}.$$

After a short computation, K_d^* can be further simplified into

$$(6.13) \quad K_d^* = \sum_{h \sim H} \sum_{m \sim X/d} C_h A_m \sum_{1 \leq k \leq K} \sum_{\substack{l < 0 \\ \gcd(l, k) = 1 \\ |l| \in \mathcal{L}(d, h, m, k)}} \sum_{n \in \mathcal{J}(d, h, m; l/k)} a(n) e(-hd^2 \gamma m^\gamma n^\gamma),$$

where

$$\mathcal{L}(d, h, m, k) = \left[\frac{hd^{1+\gamma} m^\gamma \gamma k}{Y_2^{1-\gamma}}, \frac{hd^{1+\gamma} m^\gamma \gamma k}{Y_1^{1-\gamma}} \right) \cap \left[\frac{hd^2 m^\gamma k}{N_2^{1-\gamma}}, \frac{hd^2 m^\gamma k}{N_1^{1-\gamma}} \right).$$

In the following sections, we shall show that

$$(6.14) \quad K_d^* \ll \frac{N^{1-3\eta}}{d} \left(\frac{Y}{d} \right)^{(\kappa-1)/2},$$

with a suitably chosen parameter K .

7. TRANSFORMATION OF SHORT EXPONENTIAL SUMS WITH HECKE EIGENVALUE COEFFICIENTS

In this section, we transform the short exponential sum

$$(7.1) \quad \sum_{M_1 \leq m \leq M_2} a(m)e(-tm^\gamma)$$

using Lemma 8. Note that the inner-most sum of (6.13) is of this form. We have the following

Lemma 19. *Let $\delta_1, \delta_2, \dots$ denote positive constants which may be supposed to be arbitrarily small. Let $t > 0$, $K \geq 1$, $A_1 \asymp 1$, $A_2 \asymp 1$, $l, k \in \mathbb{Z}$ with $l < 0$, $1 \leq k \leq K$ and $\gcd(l, k) = 1$. Set*

$$(7.2) \quad g(x) = -tx^\gamma,$$

where $1/2 < \gamma < 1$. Define positive real numbers M_0 , M_1 and M_2 by

$$(7.3) \quad g'(M_0) = \frac{l}{k}, \quad g'(M_1) = \frac{l}{k} - \frac{A_1}{kK} \text{ and } g'(M_2) = \frac{l}{k} + \frac{A_2}{kK}$$

and assume that $2 \leq M_1 < M_2 \leq 2M_1$. We further assume that

$$(7.4) \quad 1 \leq k \leq K \ll M_1^{1/2-\delta_1}$$

and

$$(7.5) \quad \frac{M_1^{1-\gamma+\delta_3}}{t} \ll kK \ll \frac{M_1^{1-\gamma/2-\delta_2}}{t^{1/2}} \text{ and } k^3 K \ll \frac{M_1^{3-2\gamma-\delta_2}}{t^2}.$$

Then we have

$$(7.6) \quad \begin{aligned} & \sum_{M_1 \leq m \leq M_2} a(m)e(-tm^\gamma) \\ &= i \left(\frac{t\gamma}{|l|/k} \right)^{\frac{\kappa-1/2}{2(1-\gamma)}} \sqrt{\frac{1}{2|l|(1-\gamma)}} \sum_{j=1}^2 (-1)^{j-1} \sum_{n < A_j^2 M_0 / K^2} a(n) n^{-\kappa/2+1/4} e(F_{j,k,l,t}(n)) \\ & \quad + O \left(M_1^\varepsilon \left(\frac{M_1^{\kappa/2+2-3\gamma/2}}{t^{3/2} k^{3/2} K^{5/2}} + \sqrt{\frac{k}{K}} M_1^{\kappa/2} + M_1^{\kappa/2-1/4} k^{3/4} K^{1/4} \right) \right), \end{aligned}$$

where $F_{j,k,l,t}$ is a twice-differentiable function on the interval $[1, A_j^2 M_0 K^{-2}]$ whose second derivative satisfies the asymptotic estimate

$$(7.7) \quad F''_{j,k,l,t}(x) = \frac{(-1)^j}{2x^{3/2}k} \left(\frac{t\gamma}{|l|/k} \right)^{\frac{1}{2(1-\gamma)}} \left(1 + O \left(\frac{M_1^{2(1-\gamma)}}{t^2 k^2 K^2} \right) \right).$$

Proof. We apply Lemma 8 with

$$g(z) = -tz^\gamma = -t \exp(\gamma \log z),$$

$\log z$ being the principal branch of the logarithm, M_1 and M_2 defined in (7.3) and $w(z) = 1$. We may set

$$(7.8) \quad W = 1, \quad G = tM_1^\gamma.$$

so that the conditions in (2.5) and (2.6) are satisfied (note that for (2.6) to be satisfied, the negative sign in the definition of $g(z)$ is necessary). In the following, we check that the remaining conditions in Lemma 8 are satisfied following the notations of the same. We set $r = l/k$ and note that

$$|r| \asymp GM_1^{-1}$$

by (7.3). For

$$M_0 = \left(\frac{t\gamma k}{|l|} \right)^{\frac{1}{1-\gamma}} \in (M_1, M_2),$$

we have $g'(M_0) = r$. Hence, (2.7) is satisfied. By (7.3), we have

$$M_1 = \left(\frac{t\gamma}{|l|/k + A_1/(kK)} \right)^{\frac{1}{1-\gamma}}, \text{ and } M_2 = \left(\frac{t\gamma}{|l|/k - A_2/(kK)} \right)^{\frac{1}{1-\gamma}}.$$

Using Taylor's theorem from differential calculus, we have the following estimate for m_1 as defined in (2.8).

$$m_1 = \left(\frac{t\gamma}{|l|/k} \right)^{\frac{1}{1-\gamma}} - \left(\frac{t\gamma}{|l|/k + A_1/(kK)} \right)^{\frac{1}{1-\gamma}} \asymp \frac{A_1}{kK} \left(\frac{t\gamma}{|l|/k} \right)^{\frac{1}{1-\gamma}} \frac{1}{|l|/k}.$$

We further estimate the above expression using $A_1 \asymp 1$ and

$$(7.9) \quad |r| = \frac{|l|}{k} \asymp tM_1^{\gamma-1}$$

and get

$$(7.10) \quad m_1 \asymp \frac{M_1^{2-\gamma}}{tkK}.$$

Similarly, we get

$$(7.11) \quad m_2 \asymp \frac{M_1^{2-\gamma}}{tkK}.$$

By a short calculation, we see that the condition in (7.5) is equivalent to that in (2.9) in Lemma 8 by the virtue of (7.9) and (7.10). Hence, all conditions in Lemma 8 are satisfied.

Following the notations of Lemma 8, we further have for $j = 1, 2$

$$(7.12) \quad p_{j,n}(x) = -tx^\gamma - \frac{l}{k}x + (-1)^{j-1} \left(\frac{2\sqrt{nx}}{k} - \frac{1}{8} \right)$$

and

$$(7.13) \quad n_j = \left(\frac{l}{k} - g'(M_j) \right)^2 k^2 M_j = \left(\frac{A_j}{kK} \right)^2 k^2 M_j = \frac{A_j^2}{K^2} M_j.$$

To establish (7.6), we now approximate the terms in the exponential sum appearing on the right-hand side of (2.10), where we assume that $n < n_j$. As in Lemma 8, we denote by $x_{j,n}$ the unique zero of $p'_{j,n}(x)$ in the interval (M_1, M_2) . We first observe that

$$(7.14) \quad x_{j,n} - x_{j,0} \ll M_2 - M_1 = m_1 + m_2 \ll \frac{M_1^{2-\gamma}}{tkK}$$

by (7.10) and (7.11). Hence

$$(7.15) \quad x_{j,n} = x_{j,0} \left(1 + O \left(\frac{M_1^{1-\gamma}}{kKt} \right) \right).$$

Moreover,

$$(7.16) \quad x_{j,0} = \left(\frac{t\gamma}{|l|/k} \right)^{\frac{1}{1-\gamma}} = M_0.$$

Hence, we have

$$(7.17) \quad x_{j,n}^{\kappa/2-3/4} = \left(\frac{t\gamma}{|l|/k} \right)^{\frac{\kappa/2-3/4}{1-\gamma}} \left(1 + O \left(\frac{M_1^{1-\gamma}}{kKt} \right) \right)$$

by Taylor's theorem, (7.15) and (7.16). Furthermore, using Taylor's theorem again, (7.5), (7.13), (7.15) and (7.16), we obtain

$$\begin{aligned}
(7.18) \quad p''_{j,n}(x_{j,n}) &= p''_{j,n}(x_{j,0}) \left(1 + O\left(\frac{M_1^{1-\gamma}}{kKt}\right) \right) \\
&= \left(t\gamma(1-\gamma)x_{j,0}^{\gamma-2} + (-1)^j \frac{\sqrt{n}}{2x_{j,0}^{3/2}k} \right) \left(1 + O\left(\frac{M_1^{1-\gamma}}{kKt}\right) \right) \\
&= t\gamma(1-\gamma)x_{j,0}^{\gamma-2} \left(1 + O\left(\frac{M_1^{1-\gamma}}{kKt}\right) \right)^2 \\
&= t\gamma(1-\gamma)x_{j,0}^{\gamma-2} \left(1 + O\left(\frac{M_1^{1-\gamma}}{kKt}\right) \right) \\
&= (t\gamma)^{\frac{1}{1-\gamma}} (1-\gamma) \left(\frac{|l|}{k} \right)^{\frac{2-\gamma}{1-\gamma}} \left(1 + O\left(\frac{M_1^{1-\gamma}}{kKt}\right) \right)
\end{aligned}$$

which implies that

$$(7.19) \quad p''_{j,n}(x_{j,n})^{-1/2} = (t\gamma)^{\frac{1}{2(1-\gamma)}} (1-\gamma)^{-1/2} \left(\frac{|l|}{k} \right)^{\frac{\gamma-2}{2(1-\gamma)}} \left(1 + O\left(\frac{M_1^{1-\gamma}}{kKt}\right) \right).$$

Furthermore, n_j defined in (7.13) can be approximated in the following way.

$$(7.20) \quad n_j = \frac{A_j^2}{K^2} M_0 + O\left(\frac{M_1^{2-\gamma}}{tkK^3}\right) = \frac{A_j^2}{K^2} M_0 \left(1 + O\left(\frac{M_1^{1-\gamma}}{kKt}\right) \right),$$

where we have used (7.14).

Moreover, we write

$$(7.21) \quad F_{j,k,l,t}(n) = p_{j,n}(x_{j,n}) + \frac{1}{8} - n \cdot \frac{\bar{l}}{k}.$$

Now applying (2.10) and using (7.17), (7.19), (7.20), (7.21), (7.8), (7.9), (7.10) and the fact that $a(n) \ll n^{\kappa/2-1/2+\varepsilon}$, we get the asymptotic estimate (7.6).

The remaining task is to prove the asymptotic estimate (7.7). Like in [17], we interpret n in (7.21), for a moment, as a continuous variable and aim to approximate the second derivative of $F_{j,k,l,t}(n)$ with respect to n . We cannot directly use the approximation obtained in [17] since it turns out to be not sufficient for our purposes. In the following, we refine Jutila's treatment of the said second derivative by evaluating the terms appearing in his method more precisely.

Similarly as on page 107 in [17], we have

$$\frac{dF_{j,k,l,t}(n)}{dn} = -\frac{\bar{l}}{k} + p'_{j,n}(x_{j,n}) \frac{dx_{j,n}}{dn} + (-1)^{j-1} k^{-1} n^{-1/2} x_{j,n}^{1/2} = -\frac{\bar{l}}{k} + (-1)^{j-1} k^{-1} n^{-1/2} x_{j,n}^{1/2},$$

and further (compare with (4.3.28) on page 107 in [17])

$$\frac{d^2 F_{j,k,l,t}(n)}{dn^2} = (-1)^{j-1} \frac{1}{2} k^{-1} n^{-1/2} x_{j,n}^{-1/2} \frac{dx_{j,n}}{dn} - (-1)^{j-1} \frac{1}{2} k^{-1} n^{-3/2} x_{j,n}^{1/2}.$$

We now express $dx_{j,n}/dn$ explicitly. We have

$$p'_{j,n}(x) = -t\gamma x^{\gamma-1} - \frac{l}{k} + (-1)^{j-1} \frac{\sqrt{n}}{\sqrt{x}k}$$

and hence, by the definition of $x_{j,n}$,

$$(7.22) \quad f(x_{j,n}, n) = 0,$$

where

$$f(x, n) = t\gamma kx^{\gamma-1/2} - |l|\sqrt{x} - (-1)^{j-1}\sqrt{n}.$$

By implicit differentiation, we thus get

$$(7.23) \quad \frac{dx_{j,n}}{dn} = -\frac{f_n(x_{j,n}, n)}{f_x(x_{j,n}, n)} = (-1)^{j-1} \frac{\sqrt{x_{j,n}}}{\sqrt{n} \left(t\gamma(2\gamma-1)kx_{j,n}^{\gamma-1} - |l| \right)}.$$

Hence,

$$(7.24) \quad \frac{d^2 F_{j,k,l,t}(n)}{dn^2} = \frac{1}{2kn \left(t\gamma(2\gamma-1)kx_{j,n}^{\gamma-1} - |l| \right)} - (-1)^{j-1} \frac{1}{2} k^{-1} n^{-3/2} x_{j,n}^{1/2}.$$

We now approximate the terms on the right-hand side of (7.24). Using Taylor's formula together with (7.5), (7.9), (7.15) and (7.16), we have the following asymptotic estimate for the first term.

$$(7.25) \quad \begin{aligned} \frac{1}{2kn \left(t\gamma(2\gamma-1)kx_{j,n}^{\gamma-1} - |l| \right)} &= \frac{1}{2kn \left(t\gamma(2\gamma-1)kx_{j,0}^{\gamma-1} - |l| \right)} \left(1 + O \left(\frac{M_1^{1-\gamma}}{kKt} \right) \right) \\ &= \frac{1}{4(\gamma-1)k|l|n} \left(1 + O \left(\frac{M_1^{1-\gamma}}{kKt} \right) \right) \\ &= \frac{1}{4(\gamma-1)k|l|n} + O \left(\frac{M_1^{2-2\gamma}}{t^2 k^3 K n} \right). \end{aligned}$$

For the approximation of the second term, we will need a better approximation than (7.15) for $x_{j,n}$. We set

$$g(x) = t\gamma kx^{\gamma-1/2} - |l|\sqrt{x}.$$

Then by Taylor's theorem, (7.22) and $g(x_{j,0}) = 0$, we have

$$(-1)^{j-1}\sqrt{n} = g(x_{j,n}) = g'(x_{j,0})(x_{j,n} - x_{j,0}) + O \left(\sup_{x \in (M_1, M_2)} |g''(x)| (M_2 - M_1)^2 \right)$$

which implies that

$$x_{j,n} - x_{j,0} = (-1)^{j-1} \frac{\sqrt{n}}{g'(x_{j,0})} + O(M_1^{-1}(M_2 - M_1)^2).$$

From the above, using (7.14) and (7.16), we obtain

$$(7.26) \quad x_{j,n} - x_{j,0} = (-1)^{j-1} \frac{\sqrt{nx_{j,0}}}{(\gamma-1)|l|} + O \left(\frac{M_1^{3-2\gamma}}{t^2 k^2 K^2} \right).$$

Now using Taylor's formula, (7.14) and (7.26), we obtain the following asymptotic estimate for the second term on the right-hand side of (7.24).

$$(7.27) \quad \begin{aligned} (-1)^{j-1} \frac{1}{2} k^{-1} n^{-3/2} x_{j,n}^{1/2} &= (-1)^{j-1} \frac{1}{2} k^{-1} n^{-3/2} x_{j,0}^{1/2} + (-1)^{j-1} \frac{x_{j,n} - x_{j,0}}{4kn^{3/2}x_{j,0}^{1/2}} + O \left(\frac{(M_2 - M_1)^2}{kn^{3/2}x_{j,0}^{3/2}} \right) \\ &= (-1)^{j-1} \frac{1}{2} k^{-1} n^{-3/2} x_{j,0}^{1/2} + \frac{1}{4k|l|(\gamma-1)n} + O \left(\frac{M_1^{5/2-2\gamma}}{t^2 k^3 K^2 n^{3/2}} \right). \end{aligned}$$

We note that the error term in (7.25) can be absorbed into the error term in (7.27) since $n \ll M_1/K^2$. Now combining (7.16), (7.24), (7.25) and (7.27), we get the relation (7.7). This completes the proof. \square

8. REDUCTION TO MULTI-LINEAR SUMS

In this section, we transform the sum K_d^* appearing in (6.13) into a multi-linear exponential sum with monomials. Let

$$(8.1) \quad K_d^*(Q) = \sum_{h \sim H} \sum_{m \sim X/d} C_h A_m \sum_{k \sim Q} \sum_{\substack{l < 0 \\ \gcd(l, k) = 1 \\ |l| \in \mathcal{L}(d, h, m, k)}} \sum_{n \in \mathcal{J}(d, h, m; l/k)} a(n) e(-hd^{2\gamma} m^\gamma n^\gamma)$$

be the contribution to K_d^* of the terms with $k \sim Q$. To prove (6.14), it suffices to show that

$$(8.2) \quad K_d^*(Q) \ll \frac{N^{1-4\eta}}{d} \left(\frac{Y}{d} \right)^{(\kappa-1)/2},$$

for $1 \leq Q \leq K$. A short computation using (6.9) and $XY = N$ gives that the trivial bound for $K_d^*(Q)$ is

$$K_d^*(Q) \ll \frac{Q}{K} \frac{N^{1+\varepsilon}}{d^2} H \left(\frac{Y}{d} \right)^{(\kappa-1)/2}.$$

Hence it is enough to prove (8.2) for

$$(8.3) \quad N^{-5\eta} \frac{dK}{H} \ll Q \ll K.$$

In this case, we transform the inner-most sum of (8.1) using Lemma 19 into

$$(8.4) \quad \begin{aligned} & \sum_{n \in \mathcal{J}(d, h, m; l/k)} a(n) e(-hd^{2\gamma} m^\gamma n^\gamma) \\ &= i \left(\frac{hd^{2\gamma} m^\gamma \gamma}{|l|/k} \right)^{\frac{\kappa-1/2}{2(1-\gamma)}} \sqrt{\frac{1}{2|l|(1-\gamma)}} \sum_{j=1}^2 (-1)^{j-1} \sum_{n < \mathcal{N}_j(d, h, m; l/k)} a(n) n^{-\kappa/2+1/4} e(F_{j, k, |l|, h, m, d}(n)) \\ & \quad + O \left(N^\varepsilon \left(\frac{(Y/d)^{\kappa/2+2}}{H^{3/2} N^{3\gamma/2} Q^{3/2} K^{5/2}} + \sqrt{\frac{Q}{K}} \left(\frac{Y}{d} \right)^{\kappa/2} + \left(\frac{Y}{d} \right)^{\kappa/2-1/4} Q^{3/4} K^{1/4} \right) \right), \end{aligned}$$

where

$$\mathcal{N}_j \left(d, h, m; \frac{l}{k} \right) = \frac{A_j(l/k)^2}{K^2} \left(\frac{hd^{2\gamma} m^\gamma \gamma}{|l|/k} \right)^{\frac{1}{1-\gamma}}$$

and the second derivative of the function $F_{j, k, |l|, h, m, d}$ satisfies the estimate

$$(8.5) \quad F_{j, k, |l|, h, m, d}''(x) = \frac{(-1)^j}{2x^{3/2}k} \left(\frac{hd^{2\gamma} m^\gamma \gamma}{|l|/k} \right)^{\frac{1}{2(1-\gamma)}} \left(1 + O \left(\left(\frac{Y}{HdN^\gamma QK} \right)^2 \right) \right).$$

We also require that the conditions in (7.4) and (7.5) are satisfied. It is easy to check, using (8.3), that this is the case if the following condition holds.

$$(8.6) \quad N^{6\eta} \frac{Y^{1/2}}{dN^{\gamma/2}} \ll K \ll N^{-6\eta} \min \left\{ \frac{Y^{1/2}}{H^{1/4} d^{1/2} N^{\gamma/4}}, \frac{Y^{3/4}}{H^{1/2} d^{3/4} N^{\gamma/2}} \right\}.$$

We now insert (8.4) into (8.1). The contribution to $K_d^*(Q)$ of the O -terms in (8.4) is

$$\ll N^\varepsilon E_d \left(\frac{Y}{d} \right)^{\kappa/2-1/2}$$

with

$$(8.7) \quad E_d := \frac{H^{1/2} N^{1-\gamma/2} Y^{1/2}}{K^2 d^{5/2}} + \frac{H^2 N^{1+\gamma} K^2}{Y^{3/2} d^{1/2}} + \frac{H^2 N^{1+\gamma} K^3}{Y^{7/4} d^{1/4}},$$

where we have used the facts $Q \ll K$ and $XY = N$. Hence, to establish (8.2), we require that

$$(8.8) \quad E_d \ll \frac{N^{1-5\eta}}{d}.$$

The main term takes the form

$$\sum_{h \sim H} \sum_{m \sim X/d} \sum_{k \sim Q} \sum_{\substack{l < 0 \\ \gcd(l, k) = 1 \\ |l| \in \mathcal{L}(d, h, m, k)}} \sum_{j=1}^2 \sum_{n < \mathcal{N}_j(d, h, m; l/k)} \cdots$$

We make the summation ranges for l and n independent of the other variables by using Perron's formula, Lemma 6, several times. This treatment is for convenience rather than a matter of necessity, as the application of Perron's formula enables us to avoid some summation conditions which one would otherwise encounter. We note that the contribution arising from the error term in this treatment, the O -term in Perron's formula (2.4), is negligible. Then after re-arranging the order of summations, breaking the l -range into dyadic intervals and estimating the sizes of the coefficients (where we use the Ramanujan-Petersson bound), it suffices to show that

$$(8.9) \quad \frac{Y^{3/4}}{H^{1/2} d^{3/4} N^{\gamma/2} Q^{1/2}} T_{j,d}^{\pm}(Q) \ll \frac{N^{1-5\eta}}{d}$$

with

$$(8.10) \quad T_{j,d}(Q) := \sum_{n < c_1 Y / (K^2 d)} n^{-1/4} \sum_{k \sim Q} \left| \sum_{h \sim H} \sum_{m \sim X/d} \sum_{\substack{l \sim c_2 L \\ \gcd(l, k) = 1}} \psi_{k,l,h,m} e(F_{j,k,l,h,m,d}(n)) \right|$$

in order to establish (8.2), where $j = 1, 2$, $|\psi_{k,l,h,m}| \leq 1$, c_1 and c_2 are positive constants of bounded size, and

$$(8.11) \quad L := HdN^{\gamma}Y^{-1}Q.$$

9. ESTIMATION OF THE MULTI-LINEAR SUMS

We now estimate the multi-linear sum $T_{j,d}(Q)$ defined in (8.10). It will suffice to consider the case $j = 2$. The treatment of the other case $j = 1$ is similar. First, we break the outer sum over n into dyadic intervals and denote by $T_{2,d}(Q, R)$ the contribution to $T_{2,d}(Q)$ of the n 's with $n \sim R$, which we bound in the following.

Applying the Cauchy-Schwarz inequality, and re-arranging the order of summation, we obtain

$$(9.1) \quad T_{2,d}(Q, R)^2 \ll R^{1/2} Q \sum_k \sum_{l, h, m} \sum_{\tilde{l}, \tilde{h}, \tilde{m}} \left| \sum_{n \sim R} e(F_{2,k,\tilde{l},\tilde{h},\tilde{m},d}(n) - F_{2,k,l,h,m,d}(n)) \right|.$$

We note that the O -term in (8.5) equals $1/(KL)^2$. Hence,

$$(9.2) \quad F_{2,k,l,h,m,d}''(x) = \frac{1}{2x^{3/2}} \left(\frac{hd^{2\gamma}m^{\gamma}\gamma}{l/k} \right)^{\frac{1}{2(1-\gamma)}} \frac{1}{k} (1 + O((KL)^{-2})).$$

Moreover, we have

$$(9.3) \quad 0 < \left(\frac{hd^{2\gamma}m^{\gamma}\gamma}{l/k} \right)^{\frac{1}{2(1-\gamma)}} \frac{1}{k} \leq c_3 \frac{Y^{1/2}}{d^{1/2}Q} =: U$$

for some constant $c_3 > 0$. By $\mathcal{T}(k, \Delta)$ we denote the set of all six-tuples $(l, \tilde{l}, h, \tilde{h}, m, \tilde{m})$ with $l, \tilde{l} \sim c_2 L$, $\gcd(l, k) = \gcd(\tilde{l}, k) = 1$, $h, \tilde{h} \sim H$ and $m, \tilde{m} \sim X/d$ such that

$$\left| \left(\frac{\tilde{h}d^{2\gamma}\tilde{m}^{\gamma}\gamma}{\tilde{l}/k} \right)^{\frac{1}{2(1-\gamma)}} \frac{1}{k} - \left(\frac{hd^{2\gamma}m^{\gamma}\gamma}{l/k} \right)^{\frac{1}{2(1-\gamma)}} \frac{1}{k} \right| \leq \Delta.$$

By the following lemma, $\mathcal{T}(k, \Delta)$ has essentially the expected cardinality.

Lemma 20. *Let $0 \leq \Delta \leq U$. Then*

$$|\mathcal{T}(k, \Delta)| \ll N^\epsilon \left(LHXd^{-1} + \frac{\Delta}{U} L^2 H^2 X^2 d^{-2} \right).$$

We postpone the proof of Lemma 20 to the next section. We now set

$$\Delta_0 := N^\eta U \left(\frac{1}{(KL)^2} + \frac{d}{LHX} \right).$$

Then Lemma 20 implies that

$$(9.4) \quad |\mathcal{T}(k, \Delta)| \ll N^\eta \frac{\Delta}{U} L^2 H^2 X^2 d^{-2} \text{ if } \Delta \geq \Delta_0.$$

We further set

$$\mathcal{T}'(k, \Delta) = \mathcal{T}(k, \Delta) \setminus \mathcal{T}(k, \Delta/2).$$

Then from (9.1), we deduce that

$$(9.5) \quad T_{2,d}(Q, R)^2 \ll R^{1/2} Q \sum_{k \sim Q} \left(\Sigma_1(k) + (\log N) \max_{\Delta_0 \leq \Delta \leq U} \Sigma_2(k, \Delta) \right),$$

where

$$(9.6) \quad \Sigma_1(k) := \sum_{(l, \tilde{l}, h, \tilde{h}, m, \tilde{m}) \in \mathcal{T}(k, \Delta_0)} \left| \sum_{n \sim R} e \left(F_{2,k, \tilde{l}, \tilde{h}, \tilde{m}, d}(n) - F_{2,k, l, h, m, d}(n) \right) \right|$$

and

$$(9.7) \quad \Sigma_2(k, \Delta) := \sum_{(l, \tilde{l}, h, \tilde{h}, m, \tilde{m}) \in \mathcal{T}'(k, \Delta)} \left| \sum_{n \sim R} e \left(F_{2,k, \tilde{l}, \tilde{h}, \tilde{m}, d}(n) - F_{2,k, l, h, m, d}(n) \right) \right|.$$

Using (9.4), we estimate Σ_1 trivially by

$$(9.8) \quad \Sigma_1(k) \ll N^{2\eta} (RK^{-2}H^2X^2d^{-2} + RLHXd^{-1}).$$

We now turn to the estimation of Σ_2 . If $(l, \tilde{l}, h, \tilde{h}, m, \tilde{m}) \in \mathcal{T}'(k, \Delta)$, $\Delta \geq \Delta_0$ and $x \sim R$, then

$$(9.9) \quad \left| \frac{d^2}{d^2x} \left(F_{2,k, \tilde{l}, \tilde{h}, \tilde{m}, d}(x) - F_{2,k, l, h, m, d}(x) \right) \right| \asymp \frac{\Delta}{R^{3/2}}.$$

Now (9.9), Lemma 11 and (9.4) yield the estimate

$$(9.10) \quad \Sigma_2 \ll N^\eta U^{-1} L^2 H^2 X^2 d^{-2} \left(R^{1/4} \Delta^{3/2} + R^{3/4} \Delta^{1/2} \right).$$

Combining (9.5), (9.8) and (9.10), we obtain

$$(9.11) \quad T_{2,d}(Q, R)^2 \ll N^{2\eta} Q^2 L^2 H^2 X^2 d^{-2} \left(R^{3/2} K^{-2} L^{-2} + R^{3/2} L^{-1} H^{-1} X^{-1} d + R^{3/4} U^{1/2} + R^{5/4} U^{-1/2} \right).$$

Using $R \ll Y/(dK^2)$, $Q \ll K$, (8.11), the definition of U in (9.3), and $XY = N$, we deduce from (9.11) that

$$(9.12) \quad T_{2,d}(Q) \ll N^\eta \left(QK^{-5/2} H d^{-7/4} N Y^{-1/4} + Q^{3/2} K^{-3/2} H d^{-3/4} N^{1/2+\gamma/2} Y^{-1/4} \right. \\ \left. + Q^{7/4} K^{-3/4} H^2 d^{-1/2} N^{1+\gamma} Y^{-3/2} \right).$$

A similar estimate for $T_{1,d}(Q)$ can be established in essentially the same way. Hence, (8.9) holds if

$$(9.13) \quad \frac{H^{1/2} N^{1-\gamma/2} Y^{1/2}}{K^2 d^{5/2}} + \frac{H^{1/2} N^{1/2} Y^{1/2}}{K^{1/2} d^{3/2}} + \frac{K^{1/2} H^{3/2} N^{1+\gamma/2}}{Y^{3/4} d^{5/4}} \ll \frac{N^{1-6\eta}}{d}.$$

10. PROOF OF THE SPACING LEMMA

In this section, we provide a proof of Lemma 20.

Proof. (of Lemma 20) Throughout, we assume that $l, \tilde{l} \sim c_2 L$, $h, \tilde{h} \sim H$ and $m, \tilde{m} \sim X/d$ and ignore the condition $\gcd(l, k) = \gcd(\tilde{l}, k) = 1$ in the definition of $\mathcal{T}(k, \Delta)$.

We first observe that the inequality

$$(10.1) \quad \left(\frac{\tilde{h} d^{2\gamma} \tilde{m}^{\gamma\gamma}}{\tilde{l}/k} \right)^{\frac{1}{2(1-\gamma)}} \frac{1}{k} - \left(\frac{h d^{2\gamma} m^{\gamma\gamma}}{l/k} \right)^{\frac{1}{2(1-\gamma)}} \frac{1}{k} \ll \Delta$$

holds if

$$(10.2) \quad \left(\frac{\tilde{h} d^{2\gamma} \tilde{m}^{\gamma\gamma}}{\tilde{l}/k} \right)^{\frac{1}{2(1-\gamma)}} \frac{1}{k} \left(\left(\frac{h d^{2\gamma} m^{\gamma\gamma}}{l/k} \right)^{\frac{1}{2(1-\gamma)}} \frac{1}{k} \right)^{-1} - 1 \ll \frac{\Delta}{U},$$

with the implied \ll -constant in (10.2) depending on the implied \ll -constant in (10.1). The inequality (10.2) can be simplified into

$$(10.3) \quad \left(\frac{\tilde{l}\tilde{h}}{\tilde{l}h} \right)^{\frac{1}{2(1-\gamma)}} \left(\frac{\tilde{m}}{m} \right)^{\frac{\gamma}{2(1-\gamma)}} - 1 \ll \frac{\Delta}{U}$$

which is satisfied if

$$(10.4) \quad \left(\frac{\tilde{m}}{m} \right)^{\frac{\gamma}{2(1-\gamma)}} - \left(\frac{\tilde{l}h}{l\tilde{h}} \right)^{\frac{1}{2(1-\gamma)}} \ll \frac{\Delta}{U},$$

with the implied \ll -constant in (10.4) depending on the implied \ll -constant in (10.3). Now the number of solutions to the inequality (10.4) does not exceed the product of

$$\max_{r \sim LH} d(r)^2$$

and the number of solutions to the inequality

$$(10.5) \quad \left(\frac{\tilde{m}}{m} \right)^{\frac{\gamma}{2(1-\gamma)}} - \left(\frac{\tilde{r}}{r} \right)^{\frac{1}{2(1-\gamma)}} \ll \frac{\Delta}{U},$$

where $m, \tilde{m} \sim X/d$ and $r, \tilde{r} \asymp LH$. Now using Lemma 10 and the bound $d(r) \ll r^\varepsilon$ for the divisor function, we obtain the desired result. \square

11. ESTIMATION OF K_d , SECOND METHOD

We are now ready to formulate another condition under which the desired estimate $K_d \ll N^{1-3\eta} d^{-1}$ holds. This will be favorable in the situation when Y is large.

Lemma 21. *We have*

$$(11.1) \quad K_d \ll N^{1-3\eta} d^{-1},$$

provided that $\gamma > 1/2$, $1 \leq H \leq N^{1-\gamma+\eta}$, $1 \leq d \leq 2Y$ and

$$(11.2) \quad N^{26\eta} \max \left\{ 1, \frac{H^{1/4} Y^{1/4}}{N^{\gamma/4} d^{3/4}}, \frac{Y}{N^{\gamma} d} \right\} \ll \min \left\{ \frac{Y^{1/2}}{H^{1/4} N^{\gamma/4} d^{1/2}}, \frac{Y^{3/4}}{H^{1/2} N^{\gamma/2} d^{3/4}}, \frac{Y^{3/4}}{H N^{\gamma/2} d^{1/4}}, \frac{Y^{7/12}}{H^{2/3} N^{\gamma/3} d^{1/4}}, \frac{Y^{3/2} d^{1/2}}{H^3 N^{\gamma}} \right\}.$$

Proof. According to the results in sections 6 - 9, (11.1) holds if there exists a real number $K \geq 1$ satisfying the conditions

$$(11.3) \quad K \gg N^{4\eta} \frac{Y}{N^\gamma d^2},$$

$$(11.4) \quad N^{6\eta} \frac{Y^{1/2}}{d N^{\gamma/2}} \ll K \ll N^{-6\eta} \min \left\{ \frac{Y^{1/2}}{H^{1/4} d^{1/2} N^{\gamma/4}}, \frac{Y^{3/4}}{H^{1/2} d^{3/4} N^{\gamma/2}} \right\},$$

$$(11.5) \quad \frac{H^{1/2} N^{1-\gamma/2} Y^{1/2}}{K^2 d^{5/2}} + \frac{H^2 N^{1+\gamma} K^2}{Y^{3/2} d^{1/2}} + \frac{H^2 N^{1+\gamma} K^3}{Y^{7/4} d^{1/4}} \ll \frac{N^{1-5\eta}}{d}$$

and

$$(11.6) \quad \frac{H^{1/2} N^{1-\gamma/2} Y^{1/2}}{K^2 d^{5/2}} + \frac{H^{1/2} N^{1/2} Y^{1/2}}{K^{1/2} d^{3/2}} + \frac{K^{1/2} H^{3/2} N^{1+\gamma/2}}{Y^{3/4} d^{5/4}} \ll \frac{N^{1-6\eta}}{d}.$$

The above inequalities (11.3), (11.4), (11.5) and (11.6) correspond to the inequalities (6.12), (8.6), (8.8) and (9.13), respectively. The terms in the minimum on the right-hand side of (11.4), the second and the third term on the left-hand side of (11.5) and the last term on the left-hand side of (11.6) lead to the condition

$$(11.7) \quad K \ll N^{-12\eta} \min \left\{ \frac{Y^{1/2}}{H^{1/4} N^{\gamma/4} d^{1/2}}, \frac{Y^{3/4}}{H^{1/2} N^{\gamma/2} d^{3/4}}, \frac{Y^{3/4}}{H N^{\gamma/2} d^{1/4}}, \frac{Y^{7/12}}{H^{2/3} N^{\gamma/3} d^{1/4}}, \frac{Y^{3/2} d^{1/2}}{H^3 N^\gamma} \right\}.$$

The first term on the left-hand side of (11.5) and the first two terms on the left-hand side of (11.6) lead to the condition

$$(11.8) \quad K \gg N^{12\eta} \max \left\{ \frac{H^{1/4} Y^{1/4}}{N^{\gamma/4} d^{3/4}}, \frac{HY}{Nd} \right\}.$$

Using $H \leq N^{1-\gamma+\eta}$, we observe that the lower bounds in (11.3), (11.4) and (11.8) hold if

$$(11.9) \quad K \gg N^{14\eta} \max \left\{ 1, \frac{H^{1/4} Y^{1/4}}{N^{\gamma/4} d^{3/4}}, \frac{Y}{N^\gamma d} \right\}.$$

Obviously, a number $K \geq 1$ satisfying (11.7) and (11.9) exists if (11.2) holds. This completes the proof. \square

Lemma 21 can be simplified into the following.

Lemma 22. *For every sufficiently small fixed $\eta > 0$, we have*

$$K_d \ll N^{1-3\eta} d^{-1},$$

provided that $\gamma > 13/16$, $1 \leq H \leq N^{1-\gamma+\eta}$, $1 \leq d \leq 2Y$ and

$$N^{\max\{2-4\gamma/3, 13/5-2\gamma, 6-6\gamma\}+1000\eta} \leq Y \leq 2N.$$

Proof. If $d \geq N^{3\eta} H$, then the desired estimate $K_d \ll N^{1-3\eta} d^{-1}$ follows from the trivial estimate $K_d \ll HXY/d^2 = HN/d^2$. If $d \leq N^{3\eta} H \leq N^{1-\gamma+4\eta}$ and $\gamma > 13/16$, then we use Lemma 21. In this case we calculate that (11.2) and hence $K_d \ll N^{1-3\eta} d^{-1}$ holds if

$$(11.10) \quad N^{\max\{\frac{5}{3}-\gamma, 2-\frac{4\gamma}{3}, \frac{11}{4}-\frac{5\gamma}{2}, \frac{13}{5}-2\gamma, 6-6\gamma\}+1000\eta} \leq Y \leq 2N.$$

The first term in the maximum is dominated by the second term if $\gamma < 1$, and the third term is dominated by the fourth term if $\gamma > 3/10$. This completes the proof. \square

Combining the above Lemma 22 with Lemmas 16 and 18, we arrive at the following conclusion.

Lemma 23. *For every sufficiently small fixed $\eta > 0$, we have*

$$K_d \ll N^{1-3\eta} d^{-1},$$

provided that $\gamma > 7/8$, $1 \leq H \leq N^{1-\gamma+\eta}$, $1 \leq d \leq 2Y$ and

$$N^{3-3\gamma+100\eta} \leq Y \leq 2N.$$

Proof. This follows from Lemmas 16, 18 and 22 upon noting that $\gamma > 2 - 4\gamma/3$ if $\gamma > 6/7$, $\gamma > 13/5 - 2\gamma$ if $\gamma > 13/15$, $\gamma > 6 - 6\gamma$ if $\gamma > 6/7$, and $3\gamma - 2 > 5 - 5\gamma$ if $\gamma > 7/8$. \square

12. PROOF OF THE MAIN RESULT

Proof. (of Theorems 3 and 1) We recall that Theorem 3 and hence Theorem 1, our main result, holds if (3.6) is valid for any $N \geq 1$ and $1 \leq H \leq N^{1-\gamma+\eta}$. Here γ is a fixed number in the range $7/8 < \gamma < 1$, and η is sufficiently small, which we assume in the following. Furthermore, in Lemma 12 we formulated some conditions on bilinear sums K_d and L_d under which (3.6) holds. In the following, we check that these conditions are satisfied.

We choose the parameters u , v and z in Lemma 12 as follows.

$$\begin{aligned} u &:= N^{1-\gamma+100\eta}, \\ v &:= 4N^{1/3}, \\ z &:= \left\lceil N^{\gamma/2-100\eta} \right\rceil + 1/2. \end{aligned}$$

The parameters u , v and z , so chosen, indeed satisfy the conditions in (3.7) if $\gamma > 4/5$ and η is sufficiently small. Moreover, the conditions (3.10) and (3.11) hold by Lemmas 15 and 23 since $5\gamma - 4 > 1/3$ if $\gamma > 13/15$ (the exponent which marks the limit of the method of Liu-Rivat in [21]) and $3 - 3\gamma < \gamma/2$ if $\gamma > 6/7$. This completes the proof. \square

13. NOTES

The upper bound $c < 8/7$ in Theorem 1 is due to the inequality $3\gamma - 2 > 5 - 5\gamma$, which is equivalent to $\gamma > 7/8$, in the proof of Lemma 23. If this inequality is satisfied, then the Y -ranges in Lemmas 16 and 18 overlap which is required in order to establish the Y -range in Lemma 23. For the Y -ranges in Lemmas 16 and 22 to overlap, which is also required, we only need the weaker condition $\gamma > 13/15$. In the proof of the main result in the previous section, the strongest condition occurring is also $\gamma > 13/15$. We believe that the c -range in Theorem 1 could be slightly widened by modifying the method in [25] to obtain a better lower bound for Y in Lemma 16 (which does not depend on the Hecke eigenvalues). However, we have not tried to do so since the main focus of this paper lies on the treatment of the Hecke eigenvalues.

We further note that an improvement of the lower bound in Lemma 16 would also correspond to a better upper bound for Y in Lemma 15, which would lead to a weakening of the condition $\gamma > 13/15$ occurring in section 12. However, this would be less significant since the condition $\gamma > 13/15$ already occurs in the proof of Lemma 23 and seems difficult to improve at this place. We point out that the last-mentioned condition is due to the inequality $\gamma > 13/5 - 2\gamma$ coming from the upper bound for Y in Lemma 16, which is likely to be best possible, and from the lower bound for Y in Lemma 22, which depends on our treatment of the Hecke eigenvalues.

Besides slight improvements, it would be highly desirable to prove Conjecture 1 for some c -range, making Theorem 2 unconditional for the same c -range. To this end, we need estimates for exponential sums with squares of Hecke eigenvalues (or more generally, with Fourier coefficients of Rankin-Selberg convolutions of cusp forms).

Finally, it would be interesting to generalize our result to cusp forms of arbitrary level. One would need to work out a generalization of Jutila's method to arbitrary levels for this purpose.

Acknowledgments. This project was started when S. B. visited the Division of Mathematical Sciences of Nanyang Technological University (NTU). He wishes to thank the Division for its generous financial supports and warm hospitality. S. B. further wishes to thank Jacobs University for providing excellent working conditions. L. Z. was supported by an Academic Research Fund Tier 1 Grant at NTU during this work.

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